On the variety generated by tournaments

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Tournaments

Definition. A *tournament* is a commutative groupoid satisfying $xy \in \{x, y\}$ (*conservative law*). We write $x \to y$ if $xy = x$.

Denote by $\mathcal{T}$ the variety generated by tournaments.

Theorem (1997)**.** *The variety $\mathcal{T}$ is*

(1) *locally finite,*

(2) *not finitely based, and*

(3) *inherently non-finitely generated.*

Conjecture. *Every (finite) subdirectly irreducible algebra in $\mathcal{T}$ is a tournament.*

Equations

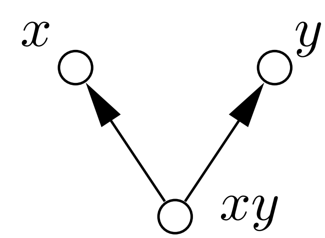
Theorem. *The following four equations form a base for the 3-variable equations of tournaments:*

(1) $xx = x$

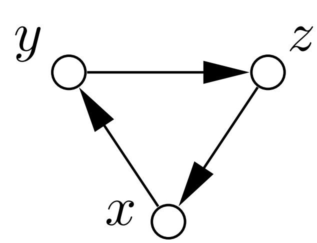
(2) $xy = yx$

(3) $(xy)x = xy$

(4) $(xy \cdot xz)(xy \cdot yz) = (xy)z$



Proposition. *If an algebra* $\mathbf{A} \in \mathcal{T}$ *does not contain any* 3*-cycles (elements* x, y, z *so that* $x \to y \to z \to x$*) then* $\mathbf{A}$ *is a semilattice.*



Partial Results

Theorem. *Every simple algebra in $\mathcal{T}$ is a tournament.*

Fact. *The conjecture holds iff for all $\mathbf{A} \in \mathcal{T}$ and for all $a, b \in A$,* $\mathrm{Cg}_{\mathbf{A}}(ab, a) \wedge \mathrm{Cg}_{\mathbf{A}}(ab, b) = 0_{\mathbf{A}}$.

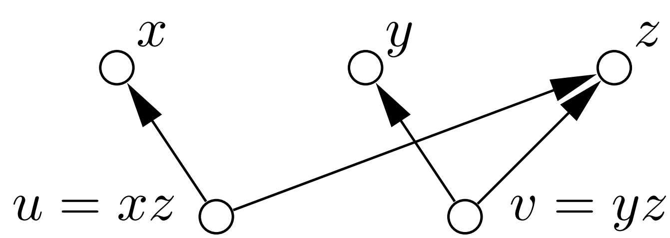
Definition. We call an algebra $\mathbf{A} \in \mathcal{T}$ *strongly connected* if for any $a, b \in A$ there exists a path $a = a_0 \to a_1 \to \ldots \to a_{n-1} = b$.

Lemma. *The conjecture holds iff every strongly connected, subdirectly irreducible algebra in $\mathcal{T}$ is a tournament.*

Basic-Translations

Let $\mathbf{A}$ be a fixed algebra in $\mathcal{T}$. For a set of pairs $S \subseteq A^2$, denote by $\mathrm{Eg}_A(S)$ the smallest equivalence relation on A containing S.

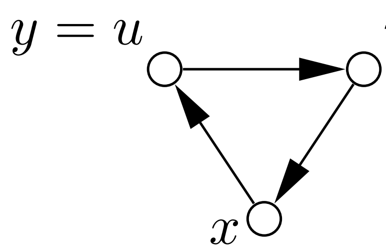
Definition. For elements $x, y, u, v \in A$, the pair $\langle u, v\rangle$ is a *basic-translation* of $\langle x, y\rangle$ if there exists $z \in A$ such that $\langle u, v\rangle = \langle xz, yz\rangle$. A *basic-ideal* is a set of pairs $I \subseteq A^2$ closed under basic-translations. For $S \subseteq A^2$, denote by $\mathrm{Ig}^{\mathrm{b}}_{\mathbf{A}}(S)$ the smallest basic-ideal containing S.

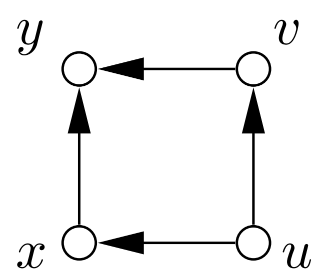


Fact. $\mathrm{Cg}_{\mathbf{A}}(S) = \mathrm{Eg}_A\,\mathrm{Ig}^{\mathrm{b}}_{\mathbf{A}}(S)$ *for all* $S \subseteq A^2$.

Cycle and Edge-Translations

Definition. For elements $x, y, u, v \in A$, the pair $\langle u, v\rangle$ is a *cycle-translation* of $\langle x, y\rangle$ if $y = u$ and $x \to y \to v \to x$. The pair $\langle u, v\rangle$ is an *edge-translation* of $\langle x, y\rangle$ if $x \to y \leftarrow v$ and $u = xv$.



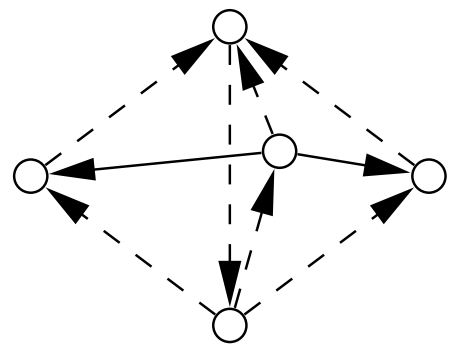


Definition. The *cycle-ideal* $\mathrm{Ig}^{\mathrm{c}}_{\mathbf{A}}(S)$, and *edge-ideal* $\mathrm{Ig}^{\mathrm{e}}_{\mathbf{A}}(S)$ generated by $S \subseteq A^2$ are the smallest sets $I \subseteq A^2$ containing S, which are closed under cycle and edge-translations, respectively.

Theorem. $\mathrm{Cg}_{\mathbf{A}}(a, b) = \mathrm{Eg}_A \,\mathrm{Ig}^{\mathrm{c}}_{\mathbf{A}}\, \mathrm{Ig}^{\mathrm{e}}_{\mathbf{A}}(a, b)$ *for all pairs of elements* $a, b \in A$ *such that* $a \to b$.

Spanning Cycle-Ideals

Definition. A *spanning cycle-ideal* of an algebra $\mathbf{A} \in \mathcal{T}$ is a cycle-ideal $I = \mathrm{Ig}^{\mathrm{c}}_{\mathbf{A}}(a, b)$ generated by some pair of elements $a \to b$, which satisfies that for all $x \in A$ there is $y \neq x$ such that $\langle x, y \rangle \in I$.



Lemma. *Each finite, strongly connected tournament has a spanning cycle-ideal.*

Lemma. *Each subdirect product of finite, strongly connected tournaments is isomorphic to a direct product of algebras, each of which has a spanning cycle-ideal.*

Spanning Cycle-Ideals (cont.)

Lemma. *Let* $\mathbf{A}, \mathbf{B} \in \mathcal{T}$ *be strongly connected algebras. Then* $\operatorname{Con}(\mathbf{A} \times \mathbf{B}) \cong \operatorname{Con} \mathbf{A} \times \operatorname{Con} \mathbf{B}$.

Lemma. *Each strongly connected algebra in* $\mathcal{T}$ *is a homomorphic image of a subdirect product of strongly connected tournaments.*

Corollary. *Each finite, strongly connected algebra in* $\mathcal{T}$ *is isomorphic to a direct product of algebras, each of which has a spanning cycle-ideal.*

Corollary. *Each finite, strongly connected, subdirectly irreducible algebra in* $\mathcal{T}$ *has a spanning cycle-ideal.*

Open Problems

Problem. *Prove the conjecture.*

Problem. *Is the variety $\mathcal{T}$ inherently non-finitely based?*

Problem. *Find a minimal list of equations, which form a base for the 4-variable equations of tournaments.*

Problem. *Describe the bottom of the lattice of subvarieties of $\mathcal{T}$.*

Conjecture. *Every subdirectly irreducible algebra in the variety determined by the 3-variable equations of tournaments is either a tournament, or contains a subalgebra isomorphic to $\mathbf{J}_3$ or $\mathbf{M}_n$ for some $n \geq 3$ (see our paper for more details).*